This process forms the $U = A^{(n)}$ portion of the matrix factorization $A = LU$. To determine the complementary lower triangular matrix L, first recall the multiplication of $A^{(k)}\mathbf{x} = \mathbf{b}^{(k)}$ by the Gaussian transformation of $M^{(k)}$ used to obtain (6.9):

$$A^{(k+1)}\mathbf{x} = M^{(k)}A^{(k)}\mathbf{x} = M^{(k)}\mathbf{b}^{(k)} = \mathbf{b}^{(k+1)},$$

where $M^{(k)}$ generates the row operations

$$(E_j - m_{j,k}E_k) \rightarrow (E_j), \quad \text{for } j = k+1, \ldots, n.$$

To reverse the effects of this transformation and return to $A^{(k)}$ requires that the operations $(E_j + m_{j,k}E_k) \rightarrow (E_j)$ be performed for each $j = k+1, \ldots, n$. This is equivalent to multiplying by the inverse of the matrix $M^{(k)}$, the matrix

$$
L^{(k)} = [M^{(k)}]^{-1} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & & & & \vdots \\
\vdots & \ddots & 0 & \ddots & & & \vdots \\
\vdots & & \vdots & m_{k+1,k} & \ddots & & \vdots \\
\vdots & & \vdots & \vdots & 0 & \ddots & 0 \\
0 & \cdots & 0 & m_{n,k} & 0 & \cdots 0 & 1
\end{bmatrix}.
$$

The lower-triangular matrix L in the factorization of A, then, is the product of the matrices $L^{(k)}$:

$$
L = L^{(1)} L^{(2)} \cdots L^{(n-1)} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
m_{21} & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
m_{n1} & \cdots & m_{n,n-1} & 1
\end{bmatrix},
$$

since the product of L with the upper-triangular matrix $U = M^{(n-1)} \cdots M^{(2)}M^{(1)}A$ gives

$$LU = L^{(1)}L^{(2)} \cdots L^{(n-3)}L^{(n-2)}L^{(n-1)} \cdot M^{(n-1)}M^{(n-2)}M^{(n-3)} \cdots M^{(2)}M^{(1)}A$$

$$= [M^{(1)}]^{-1}[M^{(2)}]^{-1} \cdots [M^{(n-2)}]^{-1}[M^{(n-1)}]^{-1} \cdot M^{(n-1)}M^{(n-2)} \cdots M^{(2)}M^{(1)}A = A.$$

Theorem

If Gaussian elimination can be performed on the linear system $A\mathbf{x} = \mathbf{b}$ without row interchanges, then the matrix A can be factored into the product of a lower-triangular matrix L and an upper-triangular matrix U, that is, $A = LU$, where $m_{ji} = a_{ji}^{(i)}/a_{ii}^{(i)}$,

$$U = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1,n}^{(n-1)} \\ 0 & \cdots & 0 & a_{nn}^{(n)} \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ m_{n1} & \cdots & m_{n,n-1} & 1 \end{bmatrix}.$$

Example

(a) Determine the LU factorization for matrix A in the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 4 \end{bmatrix}.$$

(b) Then use the factorization to solve the system

$$\begin{aligned} x_1 + x_2 \phantom{{}-x_3} + 3x_4 &= 8, \\ 2x_1 + x_2 - x_3 + x_4 &= 7, \\ 3x_1 - x_2 - x_3 + 2x_4 &= 14, \\ -x_1 + 2x_2 + 3x_3 - x_4 &= -7. \end{aligned}$$

Solution

(a) operations $(E_2 - 2E_1) \rightarrow (E_2)$

$$(E_3 - 3E_1) \rightarrow (E_3)$$

$$(E_4 - (-1)E_1) \rightarrow (E_4)$$

$$(E_3 - 4E_2) \rightarrow (E_3)$$

$$(E_4 - (-3)E_2) \rightarrow (E_4)$$

converts the system to the triangular system

$$\begin{aligned}
x_1 + x_2 \qquad\quad + \ 3x_4 &= \ \ 4, \\
-x_2 - \ x_3 - \ 5x_4 &= -7, \\
3x_3 + 13x_4 &= \ 13, \\
-13x_4 &= -13.
\end{aligned}$$

The multipliers m_{ij} and the upper triangular matrix produce the factorization

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} = LU.$$

(b) To solve

$$A\mathbf{x} = LU\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix},$$

we first introduce the substitution $\mathbf{y} = U\mathbf{x}$. Then $\mathbf{b} = L(U\mathbf{x}) = L\mathbf{y}$. That is,

$$L\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 14 \\ -7 \end{bmatrix}.$$

This system is solved for $\mathbf{y}$ by a simple forward-substitution process:

$$\begin{aligned} y_1 &= 8; & & \\ 2y_1 + y_2 &= 7, & \text{so } y_2 &= 7 - 2y_1 = -9; \\ 3y_1 + 4y_2 + y_3 &= 14, & \text{so } y_3 &= 14 - 3y_1 - 4y_2 = 26; \\ -y_1 - 3y_2 + y_4 &= -7, & \text{so } y_4 &= -7 + y_1 + 3y_2 = -26. \end{aligned}$$

We then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, the solution of the original system; that is,

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ -9 \\ 26 \\ -26 \end{bmatrix}.$$

Using backward substitution we obtain $x_4 = 2, x_3 = 0, x_2 = -1, x_1 = 3$. ■

Permutation Matrices

- The LU factorization can be applied when matrix *A* is in a form that no row interchanges are needed in the Gaussian elimination method.
- Using Permutation matrices, LU factorization method is modified for decomposition of other matrices.
- An $n \times n$ **permutation matrix** $P = [p_{ij}]$ is a matrix obtained by rearranging the rows of I_n, the identity matrix.

Illustration

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Similarly, multiplying A on the right by P interchanges the second and third columns of A. □

Suppose $k_1, \cdots, k_n$ is a permutation of the integers $1, \cdots, n$

$$p_{ij} = \begin{cases} 1, & \text{if } j = k_i, \\ 0, & \text{otherwise.} \end{cases}$$

- PA permutes the rows of A; that is,

$$PA = \begin{bmatrix} a_{k_1 1} & a_{k_1 2} & \cdots & a_{k_1 n} \\ a_{k_2 1} & a_{k_2 2} & \cdots & a_{k_2 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k_n 1} & a_{k_n 2} & \cdots & a_{k_n n} \end{bmatrix}$$

- P^{-1} exists and $P^{-1} = P^t$.

- For any nonsingular matrix A, a permutation matrix P exists for which the system

$$PA\mathbf{x} = P\mathbf{b}$$

can be solved without row interchanges. Therefore, the matrix PA can be factored into

$$PA = LU$$

This produces the factorization

$$A = P^{-1}LU = (P^tL)U$$

Example

Determine a factorization in the form $A = (P^t L)U$ for the matrix

$$A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}$$

Solution

The matrix A cannot have an LU factorization because $a_{11} = 0$.

Using operations,

$$(E_1) \leftrightarrow (E_2)$$

$$(E_3 + E_1) \rightarrow (E_3)$$

$$(E_4 - E_1) \rightarrow (E_4)$$

produces,

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Then the row interchange $(E_2) \leftrightarrow (E_4)$, followed by $(E_4 + E_3) \rightarrow (E_4)$, gives

$$U = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The permutation matrix associated with the row interchanges is

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Gaussian elimination is performed on PA using the same operations as on A, except without the row interchanges. The nonzero multipliers for PA are

$$m_{21} = 1, \quad m_{31} = -1, \quad \text{and} \quad m_{43} = -1$$

and the LU factorization of PA is

$$PA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} = LU$$

Multiplying by $P^{-1} = P^t$ produces the factorization

$$A = P^{-1}(LU) = P^t(LU) = (P^tL)U = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

HOMEWORK 8:

Exercise Set 6.5: 8 (parts b, d)

Special Types of Matrices

For these types of matrices Gaussian elimination can be performed without row interchanges.

Diagonally Dominant Matrices

The $n \times n$ matrix A is said to be **diagonally dominant** when

$$|a_{ii}| \geq \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| \quad \text{holds for each } i = 1, 2, \cdots, n.$$

A matrix is **strictly diagonally dominant** when

$$|a_{ii}| > \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| \quad \text{holds for each } i = 1, 2, \cdots, n.$$

Theorem

A strictly diagonally dominant matrix A is nonsingular. Moreover, in this case, Gaussian elimination can be performed on any linear system of the form $A\mathbf{x} = \mathbf{b}$ to obtain its unique solution without row interchanges,

Proof

suppose that a nonzero solution $\mathbf{x} = (x_i)$ to the system $A\mathbf{x} = \mathbf{0}$ exists. Let k be an index for which

$$0 < |x_k| = \max_{1\leq j\leq n} |x_j|$$

Because $\sum_{j=1}^{n} a_{ij}x_j = 0$ for each $i = 1, 2, \ldots, n$, we have, when $i = k$,

$$a_{kk}x_k = -\sum_{\substack{j=1,\\ j\neq k}}^{n} a_{kj}x_j$$

From the triangle inequality we have

$$|a_{kk}||x_k| \leq \sum_{\substack{j=1,\\ j\neq k}}^{n} |a_{kj}||x_j|, \quad \text{so} \quad |a_{kk}| \leq \sum_{\substack{j=1,\\ j\neq k}}^{n} |a_{kj}|\frac{|x_j|}{|x_k|} \leq \sum_{\substack{j=1,\\ j\neq k}}^{n} |a_{kj}|$$

This inequality contradicts the strict diagonal dominance of A.

Consequently, the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. This is shown in Theorem 6.17 to be equivalent to the nonsingularity of A.